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Solution and Ellipticity Properties of the Self-Duality Equations of Corrigan *et al.* in Eight Dimensions

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We study the two sets of self-dual Yang–Mills equations in eight dimensions proposed in 1983 by E. Corrigan *et al.* and show that one of these sets forms an elliptic system under the Coulomb gauge condition, and the other (over-determined) set can have solutions that depend at most on N arbitrary constants, where N is the dimension of the gauge group, hence the global solutions of both systems are finite dimensional. We describe a subvariety $\mathcal{P}_8$ of the skew-symmetric 8×8 matrices by an eigenvalue criterion and we show that the solutions of the elliptic equations of Corrigan *et al.* are among the maximal linear submanifolds of $\mathcal{P}_8$. We propose an eighth-order action for which the elliptic set is a maximum.

1. INTRODUCTION

The self-duality of a 2-form in four dimensions is defined to be the Hodge duality. Self-dual and anti-self-dual 2-forms can equivalently be described as eigenvectors of the completely antisymmetric fourth-rank tensor ϵ_{ijkl}. The latter approach was pursued by Corrigan *et al.* (1983), and self-dual 2-forms in n dimensions are defined as eigenvectors of a completely antisymmetric tensor invariant under a subgroup G of $SO(n)$. Then, various linear self-duality equations are obtained by specifying G. In this paper we will study two sets of equations in eight dimensions arising from invariance under $SO(7)$. These equations, denoted by *Set a* and *Set b*, are given in Section 2.

Set b consisting of 21 equations occurs in connection with other definitions of self-duality. The "strongly self-dual" 2-forms defined in Corrigan *et al.* (1983) are characterized by the property that their coefficients $\omega = (\omega_{ij})$

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with respect to an orthonormal basis satisfy the equation $\omega\omega' = \lambda I$, where λ is a nonzero constant. It is shown (Bilge *et al.*, 1996; Bilge, 1995) that this definition is equivalent to the self-duality definitions of Grossman *et al.* (1984) and Trautman (1977) and strongly self-dual 2-forms constitute an $(n^2 - n + 1)$-dimensional submanifold $\mathcal{S}_8 \cup \{0\}$ (see Definition 3.1). In eight dimensions the maximal linear submanifolds of strongly self-dual 2-forms form a six-parameter family of seven-dimensional spaces, and solutions of *Set b* are among these maximal linear submanifolds (Bilge *et al.*, 1995).

The solutions of *Set a* and *Set b* can be viewed as analogs of self-dual 2-forms in four dimensions from different aspects. The strongly self-dual 2-forms, hence the solutions of *Set b*, saturate various topological lower bounds (Bilge *et al.*, 1996; Bilge, 1995), but they form an overdetermined system. In Section 2 we show that the solutions of *Set b* for an N-dimensional gauge group depend exactly on N arbitrary constants, provided that the system is consistent. Thus *Set b* lacks the rich structure of the self-duality equations in four dimensions. On the other hand, the solutions of *Set a* do not saturate the topological lower bounds obtained in Bilge *et al.* (1996) and Bilge (1995), but these equations form an elliptic system under the Coulomb gauge condition.

In Section 3, we give an eigenvalue criterion to define a subvariety $\mathcal{P}_8$ of 8×8 skew-symmetric matrices and we show that it contains the solutions of *Set a* as a maximal linear submanifold. We give an eighth-order action whose extrema are achieved on $\mathcal{P}_8$.

2. THE SELF-DUALITY EQUATIONS OF CORRIGAN *ET AL.*

We will study the self-duality equations (3.39) and (3.40) of Corrigan *et al.* (1983), which describe a scalar field F which is an eigenvector of a fourth-rank tensor T invariant under $SO(7)$. We present below the two sets of equations corresponding to the eigenvalues 1 and -3 of T. The first set, corresponding to the eigenvalue 1, is given below. In the following, ω will denote a 2-form, and ω_{ij} will be its components with respect to an orthonormal basis.

Set a:

$$\omega_{12} + \omega_{34} + \omega_{56} + \omega_{78} = 0$$

$$\omega_{13} - \omega_{24} + \omega_{57} - \omega_{68} = 0$$

$$\omega_{14} + \omega_{23} - \omega_{67} - \omega_{58} = 0$$

$$\omega_{15} - \omega_{26} - \omega_{37} + \omega_{48} = 0$$

$$\omega_{16} + \omega_{25} + \omega_{38} + \omega_{47} = 0$$

$$\omega_{17} - \omega_{28} + \omega_{35} - \omega_{46} = 0$$
$$\omega_{18} + \omega_{27} - \omega_{36} - \omega_{45} = 0 \qquad (2.1)$$

The second set is obtained by equating the terms in each row:
Set b:

$$\omega_{12} = \omega_{34} = \omega_{56} = \omega_{78}, \ldots \qquad (2.2)$$

Note that *Set a* is the orthogonal complement of *Set b* with respect to the standard inner product on matrices, $\langle A, B\rangle = \operatorname{tr} AB'$.

2.1. The Number of Free Parameters in the Solution of *Set b*

Let F be the curvature 2-form, $F = \Sigma_{a,b} F_{ab}E_{ab}$, where the E_{ab} are basis vectors for the Lie algebra of the gauge group. Assume that each 2-form F_{ab} satisfies the equations in *Set b*, or more generally belongs to any linear submanifold of $\mathcal{S}_8 \cup \{0\}$. As these equations are overdetermined, there may not be any solutions. We recall that a topologically nontrivial solution (i.e., where F is not an exact form) is given by Grossman *et al.* (1984). Here we will show that, for an N-dimensional gauge group, if the field equations are consistent, then the solution depends at most on N arbitrary constants.

Set b represents 21 equations for the eight components of the connection 1-form. In addition, if we impose the Coulomb gauge condition, for each F_{ab} we have a system of 22 equations for eight unknowns. However, the integrability conditions of the equations for the connection 1-form become quickly very cumbersome. Thus, instead of looking at the compatibility of the differential equations for the connection, we look at the Bianchi identities, which are viewed as first-order differential equations for the curvature, i.e.,

$$dF_{ab} = A_{ac}F_{cb} - F_{ac}A_{cb} \qquad (2.3)$$

If each 2-form F_{ab} satisfies the equations in *Set b* or more generally belongs to a linear submanifold of $\mathcal{S}_8 \cup \{0\}$, it can be written as $F_{ab} = \Sigma_{i=1}^{7} \mathrm{F}^i_{ab}h_i'$ with respect to some basis $\{h_i'\}$ [one set is actually given by (2.9)]. Then

$$dF_{ab} = \sum_{i=1}^{7} \sum_{j=1}^{8} \partial_j F^i_{ab} dx^j h_i' \qquad (2.4)$$

Thus the Bianchi identities, which are 3-form equations, consist of sets of 56 algebraic equations for the 56 partial derivatives $\partial_j F^i_{ab}$ for each pair of indices (ab). It is checked that this system is nondegenerate; therefore if the gauge group is Abelian, then the Bianchi identities reduce to homogeneous equations and the F_{ab} are constants. In the non-Abelian case, the Bianchi identities form an inhomogeneous system, from which all partial derivatives of the F_{ab} are determined. Therefore, if the gauge field equations for the connection are consistent, then the resulting curvature 2-forms F_{ab} can depend at most on one arbitrary constant for each pair of indices (ab). Thus we have the following result.

Proposition 2.1. Let $F = dA - A \wedge A$, where A belongs to an N-dimensional Lie algebra and F_{ab} satisfy the equations in *Set b*. Then, if the system is compatible, F can depend at most on N arbitrary constants.

2.2. Ellipticity Properties of *Set a* and *Set b*

Recall that $F = dA - A \wedge A$, and the characteristic determinant (John, 1982) of the field equations are obtained using the linear part of this equation, i.e., $F \sim dA$. The Coulomb gauge condition is

$$\sum_i^n \partial_i A^i = 0 \tag{2.5}$$

The characteristic determinant of *Set a* together with the Coulomb gauge condition is obtained and we have the following result.

Proposition 2.2. The characteristic determinant of *Set a* together with the Coulomb gauge condition is

$$(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2 + \xi_6^2 + \xi_7^2 + \xi_8^2)^4 \tag{2.6}$$

hence the system is elliptic.

A system of elliptic equations should have as many equations as unknown. The requirement of ellipticity is the injectivity and the surjectivity of the symbol. If the symbol is injective (but not surjective), then the system is called *overdetermined elliptic*. The injectivity of the symbol leads to certain inequalities in terms of various Sobolev norms (Donaldson and Kronheimer, 1990). On the other hand, the surjectivity of the symbol guarantees the solvability of the system. Thus if a system is overdetermined elliptic, one can still use standard results from elliptic theory, provided that the existence of solutions to the overdetermined system are guaranteed.

Set b together with the Coulomb gauge condition have subsystems whose characteristic determinants are positive semidefinite. As these determinants can vanish at points where not all the ξ_i's are zero, the system is not overdetermined elliptic. Nevertheless Proposition 2.1 of the previous section implies that local solutions are always finite dimensional, hence global solutions of *Set b*, if they exist, are finite dimensional regardless of the ellipticity properties of the system.

Remark 2.3. The characteristic matrix A of *Set a* satisfies the equation $AA^t = kI$, where t denotes the transpose, I is the identity matrix, and $k = \Sigma_{i=1}^{8} \xi_i^2$. The first row of the characteristic determinant, arising from the Coulomb gauge condition, is the radial vector, hence the remaining seven rows represent tangent vector fields to S^7. Since S^3 and S^7 are the only parallelizable spheres, the equations in *Set a* are unique analogs of the self-duality equations in four dimensions, as already noted in Corrigan *et al.* (1983).

2.3. An Alternative Derivation of *Set a* and *Set b*

We recall that squares of strongly-self dual 2-forms are self-dual in the Hodge sense (Bilge *et al.*, 1996) and the maximal linear subspaces of strongly self-dual 2-forms are a six-parameter family of seven-dimensional spaces. In this section we will obtain analogs of Eqs. (2.2) that will be used in Section 3. Similar equations are also obtained in Bilge *et al.* (1995).

We fix a nondegenerate 2-form $h_1' = e_{12} + \alpha e_{34} + \beta e_{56} + \gamma e_{78}$, and we consider the 2-forms $h_j' = e_{1(j+1)} + \kappa_j'$ for $j = 2, \ldots, 7$ such that the κ_j' do not involve e_1 and e_{j+1}. The requirement that $(h_1' + h_j')^2$ be self-dual gives linear equations for the components of the h_j'. Once these equations are solved, the nonlinear equations obtained from the self-duality of $(h_i' + h_j'^2$ for $i \neq 1$ can be solved easily and we obtain the following result.

Proposition 2.4. Let $h_1' = e_{12} + \alpha e_{34} + \beta e_{56} + \gamma e_{78}$, and h_j', $j = 2, \ldots, 7$, be of the form $h_j' = e_{1(j+1)} + \kappa_j'$, such that $\langle e_1, \kappa_j' \rangle = \langle e_{j+1}, \kappa_j' \rangle = 0$. If the 4-forms $(h_i' + h_j')^2$ are self-dual for all i, j, then the h_i' are

$$
\begin{aligned}
h_1' &= e_{12} + \beta\gamma e_{34} + \beta e_{56} + \gamma e_{78} \\
h_2' &= e_{13} - \beta\gamma e_{24} + \beta c' e_{57} - \beta c e_{58} - \gamma c e_{67} - \gamma c' e_{68} \\
h_3' &= e_{14} + \beta\gamma e_{23} - c e_{57} - c' e_{58} - \beta\gamma c' e_{67} + \beta\gamma c e_{68} \\
h_4' &= e_{15} - \beta e_{26} - \beta c' e_{37} + \beta c e_{38} + c e_{47} + c' e_{48} \\
h_5' &= e_{16} + \beta e_{25} + \gamma c e_{37} + \gamma c' e_{38} + \beta\gamma c' e_{47} - \beta\gamma c e_{48} \\
h_6' &= e_{17} - \gamma e_{28} + \beta c' e_{35} - \gamma c e_{36} - c e_{45} - \beta\gamma c' e_{46} \\
h_7' &= e_{18} + \gamma e_{27} - \beta c e_{35} - \gamma c' e_{36} - c' e_{45} + \beta\gamma c e_{46}
\end{aligned}
\tag{2.7}
$$

where $\beta^2 = \gamma^2 = c^2 + c'^2 = 1$.

Thus, depending on the possible choices for β and γ, we have four sets of seven equations parametrized by c and c'. We denote these forms by h_i',

k'_i, m'_i, and n'_i corresponding, respectively, to the cases ($\beta = 1, \gamma = 1$), ($\beta = 1, \gamma = -1$), ($\beta = -1, \gamma = 1$), and ($\beta = -1, \gamma = -1$).

The set consisting of the 28 forms thus obtained is, however, linearly dependent for any c and c'. To retain similarity with (2.2) we set $c' = 1$ and $c = 0$, and we obtain the following linear submanifolds of $\mathcal{S}_8 \cup \{0\}$:

$$\begin{aligned} B^{++} &= \text{span}\{h'_1, h'_2, h'_3, h'_4, h'_5, h'_6, h'_7\} \\ B^{+-} &= \text{span}\{k'_1, k'_2, k'_3, k'_4, k'_5, k'_6, k'_7\} \\ B^{-+} &= \text{span}\{m'_1, m'_2, k'_3, m'_4, m'_5, m'_6, k'_7\} \\ B^{--} &= \text{span}\{n'_1, n'_2, h'_3, m'_4, n'_5, n'_6, k'_7\} \end{aligned} \tag{2.8}$$

A basis for 2-forms on R^8 can be obtained by adding

$$\begin{aligned} p'_1 &= e_{14} - e_{23} + e_{58} - e_{67} \\ p'_2 &= e_{14} + e_{23} + e_{58} + e_{67} \\ p'_3 &= e_{15} + e_{26} - e_{37} - e_{48} \\ p'_4 &= e_{15} - e_{26} + e_{37} - e_{48} \\ p'_5 &= e_{18} + e_{27} + e_{36} + e_{45} \\ p'_6 &= e_{18} - e_{27} - e_{36} + e_{45} \end{aligned} \tag{2.9}$$

to the 2-forms in (2.8).

The analog of the equations in *Set b* are obtained by restricting ω to the subspaces in (2.8). Similarly the analogs of *Set a* are obtained by taking orthogonal complements. The coefficients of ω with respect to the basis consisting of the h'_i, k'_i, m'_i, n'_i, and p'_i will be denoted by the same symbols without prime.

3. AN EIGENVALUE CHARACTERIZATION OF *SET a* AND AN ACTION DENSITY

We recall the following definition given in Bilge *et al.* (1996).

Definition 3.1. Let ω be a 2-form in $2n$ dimensions, with components ω_{ij} with respect to an orthonormal basis. The 2-form ω is called *strongly self-dual* if the absolute values of the eigenvalues of the matrix ω_{ij} are equal. The nonzero strongly self-dual 2-forms belong to a 13-dimensional submanifold $\mathcal{S}_8$, and the solutions of *Set b* are among the maximal linear submanifolds of $\mathcal{S}_8 \cup \{0\}$ (Bilge *et al.*, 1995).

We will define below a subvariety $\mathcal{P}_8$ which contains the solutions of *Set a* as a maximal linear submanifold.

Let the eigenvalues of the matrix ω_{ij} be $\pm i\lambda_k$, $k = 1, \ldots, 4$, and define q_j to be the jth elementary symmetric function of the λ_k^2. Then

$$(\omega, \omega) = s_2 = 4q_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2$$

$$\frac{1}{2^2}(\omega^2, \omega^2) = s_4 = 6q_2 = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_1^2\lambda_4^2 + \lambda_2^2\lambda_3^2 + \lambda_2^2\lambda_4^2 + \lambda_3^2\lambda_4^2$$

$$\frac{1}{6^2}(\omega^3, \omega^3) = s_6 = 4q_3 = \lambda_1^2\lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_2^2\lambda_4^2 + \lambda_1^2\lambda_3^2\lambda_4^2 + \lambda_2^2\lambda_3^2\lambda_4^2$$

$$\frac{1}{24^2}(\omega^4, \omega^4) = s_8 = q_4 = \lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2 \tag{3.1}$$

We have the inequalities

$$q_1^2 \geq q_2 \geq \sqrt{q_4} \tag{3.2}$$

the equalities being saturated iff all the eigenvalues are equal (Marcus and Minc, 1996), i.e., for the strongly self-dual forms. This corresponds to the case where the quantities

$$A = (\omega, \omega)^2 - \tfrac{2}{3}(\omega^2, \omega^2) \tag{3.3}$$

$$B = (\omega^2, \omega^2) - (\omega^4, \omega^4)^{1/2}$$

vanish. Proposition 3.2 below implies that the quantity

$$\Phi = A + \tfrac{1}{3}B = (\omega, \omega)^2 - \tfrac{1}{3}(\omega^2, \omega^2) - \tfrac{1}{3}(\omega^4, \omega^4)^{1/2} \tag{3.4}$$

is a measure of the power of the anti-self-dual part of ω.

Proposition 3.2. Let $(\omega^{2+}, \omega^{2+}) \geq (\omega^{2-}, \omega^{2-})$, and $\Phi = (\omega, \omega)^2 - \alpha(\omega^{2+}, \omega^{2+})$, where $\omega^{2\pm}$ denote the self-dual and anti-self-dual parts of ω^2. Then, max α such that Φ is nonnegative for all ω is $\alpha = 2/3$.

Proof. If $(\omega^{2+}, \omega^{2+}) \geq (\omega^{2-}, \omega^{2-})$, then $(\omega^4, \omega^4)^{1/2} = *\omega^4 = (\omega^{2+}, \omega^{2+}) - (\omega^{2-}, \omega^{2-})$. From the inequalities (3.2), it can be seen that $\alpha \leq 2/3$. On the other hand, the equality is attained for $\omega \in \mathcal{S}_8$, hence $\alpha = 2/3$. ■

It is an elementary fact that the product $\frac{1}{3}AB$, under the constraint $A + \frac{1}{3}B =$ const, is maximized for $\Psi = A - \frac{1}{3}B = 0$ and minimized for $A = 0$ or $B = 0$. The condition $A - \frac{1}{3}B = 0$ gives

$$\Psi = (\omega, \omega)^2 - (\omega^2, \omega^2) + \tfrac{1}{3}(\omega^4, \omega^4)^{1/2} = 0 \tag{3.5}$$

Thus we have the following result.

Proposition 3.3. Let $\Phi = (\omega, \omega)^2 - \frac{1}{3}(\omega^2, \omega^2) - \frac{1}{3}(\omega^4, \omega^4)^{1/2}$ be fixed and $(\omega, \omega)^2 - \frac{2}{3}(\omega^2, \omega^2)$ be nonzero. Then the quantity $[(\omega, \omega)^2 - \frac{2}{3}(\omega^2, \omega^2)][(\omega^2, \omega^2) - (\omega^4, \omega^4)^{1/2}]$ is maximal for $\Psi = (\omega, \omega)^2 - (\omega^2, \omega^2) + \frac{1}{3}(\omega^4, \omega^4)^{1/2} = 0$.

The expression of Ψ in terms of the ω_{ij} is very complicated. However, it reduces to a relatively simple form under a change of parameters. If we reparametrize the eigenvalues as

$$\begin{aligned}\epsilon_1 &= (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)\\ \epsilon_2 &= (\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)\\ \epsilon_3 &= (\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4)\\ \epsilon_4 &= (\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)\end{aligned} \tag{3.6}$$

then Ψ reduces to

$$\Psi = (\omega, \omega)^2 - (\omega^2, \omega^2) + \tfrac{1}{3}(\omega^4, \omega^4)^{1/2} = \epsilon_1\epsilon_2\epsilon_3\epsilon_4 \tag{3.7}$$

We note that we could obtain a similar decomposition for $(\omega, \omega)^2 - (\omega^2, \omega^2) - \frac{1}{3}(\omega^4, \omega^4)^{1/2}$ if we define the e_i with an odd number of minus signs.

The equality of the λ_i corresponds to the case where any three of the ϵ_i are zero. The appropriate nonlinear set containing solutions of *Set a* is thus the set where only one of the ϵ_i is zero. The explicit expression of Ψ when ω is written with respect to the basis given in (2.10) and (2.11) is

$$\begin{aligned}\Psi = {} & h_1[k_1(m_1n_1 + m_4p_3 + m_5n_5) + k_2(n_1m_2 - m_4p_6 + m_5n_6)\\ & + k_3(n_1p_1 - p_3n_6 - n_5p_6) + k_6(m_1n_6 + m_4p_1 - n_5m_2)\\ & + k_7(m_1p_6 + p_3m_2 + m_5p_1)]\\ & + h_2[k_1(m_1n_2 + m_4p_5 + n_5m_6) + k_2(m_4p_4 + m_2n_2 + n_6m_6)\\ & + k_3(n_5p_4 - n_6p_5 + p_1n_2) + k_5(-m_1n_6 - m_4p_1 + n_5m_2)\\ & + k_7(-m_1p_4 + m_2p_5 + p_1m_6)]\\ & + h_3[k_1(m_1p_2 + p_3m_6 - m_5p_5) + k_2(-m_5p_4 + m_2p_2 - p_6m_6)\\ & + k_3(p_3p_4 + p_6p_5 + p_1p_2) + k_5(m_1p_6 + p_3m_2 + m_5p_1)\\ & + k_6(-m_1p_4 + m_2p_5 + p_1m_6)]\\ & + h_4[m_1(n_1p_4 + p_6n_2 + n_6p_2) + m_2(-n_1p_5 + p_3n_2 - n_5p_2)\\ & + m_4(p_3p_4 + p_6p_5 + p_1p_2) + m_5(n_5p_4 - n_6p_5 + p_1n_2)\\ & + m_6(-n_1p_1 + p_3n_6 + n_5p_6)]\end{aligned}$$

$$
\begin{aligned}
&+ h_5[k_2(-n_1m_6 + m_4p_2 + m_5n_2) + k_3(n_1p_5 - p_3n_2 + n_5p_2) \\
&+ k_5(m_1n_1 + m_4p_3 + m_5n_5) + k_6(m_1n_2 + m_4p_5 + n_5m_6) \\
&+ k_7(-m_1p_2 - p_3m_6 + m_5p_5)] \\
&+ h_6[k_1(n_1m_6 - m_4p_2 - m_5n_2) + k_3(n_1p_4 + p_6n_2 + n_6p_2) \\
&+ k_5(n_1m_2 - m_4p_6 + m_5n_6) + k_6(m_4p_4 + m_2n_2 + n_6m_6) \\
&+ k_7(m_5p_4 - m_2p_2 + p_6m_6)] \\
&+ h_7[k_1(n_1p_5 - p_3n_2 + n_5p_2) + k_2(n_1p_4 + p_6n_2 + n_6p_2) \\
&+ k_5(-n_1p_1 + p_3n_6 + n_5p_6) + k_6(-n_5p_4 + n_6p_5 - p_1n_2) \\
&+ k_7(p_3p_4 + p_6p_5 + p_1p_2)].
\end{aligned} \tag{3.8}
$$

From (3.8), it can be seen that $\Psi = 0$ both on *Set a*, where all h_i are zero, and on *Set b*, where all the parameters except the h_i are zero. Actually, Ψ vanishes on the complement of each of the subspaces in (2.10), which are 21-dimensional linear submanifolds of $\mathcal{P}_8$. By assigning arbitrary values to the remaining parameters, it can be seen that these 21-dimensional submanifolds are maximal. Hence solutions of *Set a* (and their analogs) are among the maximal linear submanifolds of $\mathcal{P}_8$.

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